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Iterative Solution of Volterra Integral Equations Using Clenshaw–Curtis Quadrature

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A method for the direct iterative solution of Volterra integral equations is discussed which involves a recursive numerical integration scheme based on the procedure of Clenshaw and Curtis. The main illustrative example considered is the integral equation for neutron slowing down in an infinite non-absorbing moderating medium. Further examples are also considered and the accuracy and efficiency of the method is tested and comparison made with the results obtained from alternative prescriptions, where applicable.

1. INTRODUCTION

Recursive numerical integration schemes have recently been applied to obtain variational solutions to eigenvalue problems arising from certain integral equations. In particular, the one-electron equations of the Hartree–Fock self-consistent field model for the energies of atomic systems have been converted to integral equations and the eigenvalues obtained by use of the variation–iteration method [1, 2]. The natural iteration provided by the use of integral operators complements the usual minimum energy variational principle and accurate eigenvalues were obtained very quickly by the procedure. The numerical integration scheme used was based on the well-known procedure of Clenshaw and Curtis [3]. As pointed out in [1] this method has certain advantages in the present context over many alternative schemes. Thus: (i) high accuracy is attainable in many cases—Gauss quadrature precision being regularly approached; (ii) the method is adaptive in that, when the order of the quadrature is doubled, the previously computed function values may be re-utilized; (iii) continuous monitoring of the quadrature is possible, since local error estimates are automatically generated.

The success of the method for eigenvalue problems prompted the investigation of its application to the direct solution of integral equations arising in certain physical problems by the classical Neumann series or successive approximations approach. Previous workers on these lines (see Delves and Walsh [4]) have mainly used fairly simple quadrature procedures (usually in an algebraic context) with resulting limitations in accuracy. It is therefore of interest to investigate the efficiency of the present fully automatic method in practical situations. There are, of course,

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convergence problems associated with the use of the Neumann series, particularly with Fredholm integral equations. These difficulties may sometimes be alleviated by decomposition of the operators (with physical knowledge of the system often providing the key) as, for example, in the work of Allen and Wing [5] on the transport equation. However, this is seen as a separate issue and the present work concentrates on the feasibility of the numerical integration schemes in the context of Volterra integral equations of the second kind, where no such convergence problems arise. Integral equations of this type with convolution or displacement kernels arise naturally in many varied physical applications. In addition, initial value differential equation problems may be reformulated as Volterra integral equations, the conversion being carried out using either the classical approach of Fubini [6] or the functional analysis methods described by Rall [7]. In many such instances the integral equations arising are non-linear and the conventional algebraic approach [4] is not applicable. Hence, an investigation of the direct iterative solution is of particular interest here, especially in cases where the numerical solution of the corresponding differential equation by, say, standard Runge–Kutta methods is available for comparison.

In the present paper the efficiency of the Clenshaw–Curtis quadrature approach is examined critically and comparisons made with alternative methods of solution for several illustrative examples.

The main example considered arises in neutron slowing down theory, the classical analytical solution being available for a check on accuracy. Of particular interest here is the fact that the solution possesses a finite discontinuity, thereby providing a very stringent test of any quadrature scheme. In addition, an alternative form of the integral equation is available from the physics of the situation and a comparison of the efficiency of the suggested numerical solution as applied to both forms is of interest. Comparative information regarding a measure of the convergence of the Neumann series is provided here. Comparison with the conventional algebraic approach is also carried out since in this case the integral equation is linear.

Another representative example involving a certain convolution kernel is considered with various linear and non-linear forms of the corresponding Volterra equations. Comparison is again made with standard algebraic methods of solution.

The final example considered originates as the Lane–Emden initial value differential equation which describes the thermal behaviour of spherical clouds of gas in gravitational equilibrium [8]. The resulting integral equation is non-linear and comparison is possible here with the conventional Runge–Kutta solution of the parent differential equation.

2. Integral Equations for the Slowing Down of Neutrons

The mechanics of the slowing down of neutrons by elastic scattering in an infinite non-absorbing moderating medium is considered in detail by Weinberg and Wigner [9]. Basically, Q_0 source neutrons per unit volume at energy E_0 are slowed down to

energy E as a result of successive collisions with moderating nuclei of mass A. On introducing the lethargy variable u by means of

$$u = \ln(E_0/E) \tag{1}$$

it is seen that the collision density $F(u)$ is given by the integral equation

$$F(u) = \gamma Q_0 \exp(-u) + \gamma \int_0^u \exp(u' - u) F(u')\, du' \tag{2}$$

when $0 < u < \varepsilon$
and by the equation

$$F(u) = \gamma \int_{u-\epsilon}^u \exp(u' - u) F(u')\, du' \tag{3}$$

when $u > \varepsilon$.
In these equations

$$\gamma = (1 - \alpha)^{-1}, \tag{4}$$

where

$$\alpha = (A - 1)^2/(A + 1)^2 \tag{5}$$

and

$$\varepsilon = -\ln \alpha \tag{6}$$

represents the maximum lethargy gain per collision.

It will be observed that there is a discontinuity of magnitude βQ_0, where

$$\beta = \alpha/(1 - \alpha) \tag{7}$$

at $u = \varepsilon$. This arises since the source neutrons may contribute directly to the collision density in the lethargy range $0 < u < \varepsilon$, but not to the range $u > \varepsilon$.

The solution to Eq. (2) is easily obtained by differentiation and yields

$$F(u) = \gamma Q_0 \exp(\beta u), \qquad 0 < u < \varepsilon. \tag{8}$$

Equation (3) is much more difficult to treat and the classical solution was obtained by Placzek [10] by a recurrence relation approach involving the so-called Placzek functions. An alternative, perhaps more convenient, solution was obtained by Teichmann [11] and by Eidelman [12] by means of the Laplace Transform Convolution Theorem, the result being in the form

$$F(u) = \gamma Q_0 \exp(\beta u) + \gamma Q_0 \sum_{k=1}^{\infty} (-\beta)^k \exp(\beta z)[z^k/k! + z^{k-1}/(k-1)!]\, H(z), \tag{9}$$

where

$$z = u - k\varepsilon \tag{10}$$

and the Heaviside function $H(z)$ is defined by

$$\begin{aligned} H(z) &= 1, \qquad z \geqslant 0 \\ &= 0, \qquad z < 0. \end{aligned} \tag{11}$$

The analytical solution (9) is thus available to test the accuracy of the attempted direct numerical solution of Eqs. (2) and (3).

It is of interest to note that the basic Eqs. (2) and (3) may be re-cast into an alternative form. Thus, the slowing down density $q(E)$ representing the number of neutrons slowing down past energy E is introduced, [9] or [13], and on noting that, in the absence of absorbtion or leakage, $q(E)$ must be equal to the source density Q_0, it is easy to show that (2) and (3) are replaced by the simpler forms

$$F(u) = \gamma Q_0 + \beta \int_0^u F(u')\, du', \qquad 0 < u < \varepsilon, \tag{12}$$

and

$$F(u) = Q_0 + \beta \int_{u-\epsilon}^u F(u')\, du', \qquad u > \varepsilon. \tag{13}$$

The direct solution of these alternative forms (12) and (13) is therefore compared with the solution of the classical forms (2) and (3) from the point of view of computational ease and efficiency. It should be pointed out that forms equivalent to (12) and (13) have also been derived by Placzek [10]. Note also that the asymptotic collision density is readily available from (13) and yields

$$F \sim Q_0/\xi, \qquad u \to \infty, \tag{14}$$

where

$$\xi = 1 - \beta\varepsilon. \tag{15}$$

In practice this limiting value is attained rapidly for all values of A, so that after about three collisions the collision density is constant to within 1%, even for the largest values of A.

3. Other Examples of Integral Equations

The integral equation

$$\phi(x) = (x+2)^{-2} - \int_0^x G(\phi(t))(x - t + 2)^{-2}\, dt \tag{16}$$

was considered for purposes of illustration, three particular cases of the function G being studied, namely:

(i) $G(\phi) = 2\phi$,

(ii) $G(\phi) = 2\phi^2$,

(iii) $G(\phi) = \exp(-\phi)$.

The second and third cases provide examples of the non-linear variety.

The final example considered arises from the Lane–Emden differential equation

$$\ddot{x} + \frac{2}{t}\dot{x} + x^v = 0 \tag{17}$$

subject to $x = 1$, $\dot{x} = 0$ at $t = 0$.

Simple analytical solutions are available here in the cases $v = 0$, 1 and 5 and numerical solutions are available in tabular form for other values of v; see [8] for comparison purposes. Fubini's variation of parameters method produces the integral equation

$$x(t) = 1 + \frac{1}{t}\int_0^t s(s-t)\, x^v(s)\, ds \tag{18}$$

the non-linear term having been isolated.

4. Numerical Integration Formulae

The integral equations of the previous sections may be expressed in the general form

$$F(u) = f(u) + \mathbf{K}F(u) \tag{19}$$

where $f(u)$ is a given function, $F(u)$ is the solution to be determined and the integral operator $\mathbf{K}$ is defined by

$$\mathbf{K}F(u) = \int_v^u K(u, u')\, F(u')\, du'. \tag{20}$$

In this definition $K(u, u')$ is the kernel of the integral equation and v, the lower limit of integration, is usually zero but, in Eqs. (3) and (13), for example, it is $u - \varepsilon$ if $u > \varepsilon$. In the case of the non-linear equations of Section 3, $F(u)$ in Eq. (20) is generalized to $G(F(u))$, a non-linear function of $F(u)$.

The solution is obtained by constructing the iterative sequence of functions $\{F^{(i)}(u)\}$ generated by the recurrence relation

$$F^{(i)}(u) = f(u) + \mathbf{K}F^{(i-1)}(u) \qquad (i \geqslant 1) \tag{21}$$

with

$$F^{(0)}(u) = f(u). \tag{22}$$

The terms in this classical Neumann series are generated by recursive use of the integral operator **K**, the Clenshaw–Curtis quadrature prescription being used to compute $F^{(i)}(u)$ at specified grid-points. The iteration is continued to order $i = M$ which is such that

$$|F^{(M)}(u) - F^{(M-1)}(u)| < E\,|F^{(M)}(u)|, \tag{23}$$

where E is some specified accuracy. In the non-linear cases $F^{(i)}(u)$ is replaced by $G(F^{(i)}(u))$.

The presence of the discontinuity in the neutron slowing down equation results in the application of the quadrature rule being non-standard. This situation is therefore now described in some detail. The application to the equations of Section 3 is much more straightforward and involves only the standard Volterra range $[0, u]$ and no further details need be given.

Basically, the range of integration is sub-divided into a number of sub-intervals $R_l = [u_{l-1}, u_l]$ with $l = 1, 2, 3, \ldots$. A linear transformation maps each interval on to $[-1, 1]$ and grid-points are allocated according to the formula

$$u = u_{l,s} = \frac{1}{2}(u_l + u_{l-1}) + \frac{1}{2}(u_l - u_{l-1}) \cos\frac{\pi s}{n_l} \qquad (0 \leqslant s \leqslant n_l), \tag{24}$$

where $(n_l + 1)$ represents the number of quadrature points in the interval R_l.

In the neutron problem, it was logical to choose the sub-interval boundaries to correspond to the successive collision density lethargy intervals $[0, \varepsilon]$, $[\varepsilon, 2\varepsilon]$, $[2\varepsilon, 3\varepsilon]$,..., the maximum lethargy considered being 6ε, where the collision density has sensibly attained its constant asymptotic form. Hence

$$u_l = l\varepsilon, \qquad l = 0, 1, 2, \ldots, 6. \tag{25}$$

The basic integrations required in (21) may be expressed in the general form

$$\int_v^u \psi(u)\, du \tag{26}$$

with the grid-points specified by Eq. (24).

It will be observed that the range of integration $[u - \varepsilon, u]$ for $u > \varepsilon$ actually crosses the boundary u_{l-1} between regions R_l and R_{l-1} for $l \geqslant 2$ and hence it is advantageous to consider the sub-divided ranges separately according to

$$[v, u] = [u_{l,s} - \varepsilon, u_{l,s}] = [u_{l,s} - \varepsilon, u_{l-1}] + [u_{l-1}, u_{l,s}]. \tag{27}$$

This is indeed necessary for the case $l = 2$ since there is a discontinuity at the point $u_1 = \varepsilon$.

Consequently, for the integral over $[u_{l-1}, u_{l,s}]$ the grid-points (24) are used and the quadrature formula may be written as

$$\int_{u_{l-1}}^{u_{l,s}} \psi(u)\,du \cong \frac{1}{2}(u_l - u_{l-1}) \sum_{i=1}^{n+1} b_i \left[\cos\frac{\pi i s}{n} - (-1)^i\right] \tag{28}$$

with $n = n_l$. The coefficients b_i are related to the Clenshaw–Curtis coefficients

$$a_i = \frac{2}{n} \sum_{j=0}^{n}{}'' \psi(u_{l,j}) \cos\frac{\pi i j}{n} \tag{29}$$

by means of the relation

$$b_i = (a_{i-1} - a_{i+1})/(2i), \qquad i = 1, 2, \ldots, (n-2). \tag{30}$$

The last three coefficients are given by the special formulae

$$\begin{aligned} b_{n-1} &= \tfrac{1}{2}(a_{n-2} - \tfrac{1}{2}a_n)/(n-1), \\ b_n &= a_{n-1}/(2n), \\ b_{n+1} &= \tfrac{1}{4}a_n/(n+1). \end{aligned} \tag{31}$$

In (29) the double primes denote that the first and the last terms in the summation are to be halved. (See [1, 3].)

The quadrature formulae for the other interval is easily shown to be

$$\int_{u_{l,s}-\epsilon}^{u_{l-1}} \psi(u)\,du = \frac{1}{2}(u_{l-1} - u_{l-2}) \sum_{i=1}^{m+1} b_i' \left[1 - \cos\frac{\pi i s}{n}\right] \tag{32}$$

with $m = n_{l-1}$. The quadrature coefficients b_i' are related to

$$a_i' = \frac{2}{m} \sum_{j=0}^{m}{}'' \psi(u_{l-1,j}) \cos\frac{\pi i j}{m} \tag{33}$$

by relations of the form (30) and (31) with $n = n_l$ replaced by $m = n_{l-1}$ and the grid-points in R_{l-1} are specified by

$$u_{l-1,j} = \frac{1}{2}(u_{l-1} + u_{l-2}) + \frac{1}{2}(u_{l-1} - u_{l-2}) \cos\frac{\pi j}{n_{l-1}} \qquad (0 \leqslant j \leqslant n_{l-1}). \tag{34}$$

It will be observed that the expression $\cos(\pi i s/n)$ with $n = n_l$ occurs in *both* quadrature formulae, it is only the quadrature coefficients b_i which change form.

For the first interval $R_1 = [0, \varepsilon]$ we have the specially simple case

$$\int_0^u \psi(u)\,du \tag{35}$$

with grid-points given by

$$u = u_{1,s} = \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \cos\frac{\pi s}{n_1}, \qquad 0 \leqslant s \leqslant n_1. \tag{36}$$

The quadrature prescription is

$$\int_0^{u_{1,s}} \psi(u)\, du = \frac{1}{2}\varepsilon \sum_{i=1}^{n_1+1} b_i \left[\cos\frac{\pi i s}{n_1} - (-1)^i\right]$$

with

$$a_i = \frac{2}{n_1} \sum_{j=0}^{n_1}{}'' \psi(u_{1,j}) \cos\frac{\pi i j}{n_1}. \tag{37}$$

Only integrals of the form (35) appear in the standard Volterra examples of section 3 and therefore only the simpler prescription of equations (36) and (37) are required there. The number of quadrature grid-points is chosen to ensure that the local errors generated by the numerical integration procedure are less than the tolerance E imposed on the iteration cycle. These local quadrature errors may be monitored continuously by observing the successive values of the b_i coefficients in the manner suggested by Clenshaw and Curtis. Further discussion of the error procedure is available in the work of O'Hara and Smith [14] and Gentleman [15]. It is thus possible to produce a fully automatic version of the quadrature procedure which utilizes the adaptive nature of the Clenshaw–Curtis grid-points (24) to economize the number of function evaluations required in a given region. The usual procedure is to double the number of quadrature points used, adopting $n = 4, 8, 16, \ldots$ successively until the b_i error criterion is satisfied.

5. Results and Discussion

(a) *Neutron Slowing Down*

The slowing down densities are computed from the basic Eqs (2) and (3) or the alternative forms (12) and (13) over the lethargy range $0 \leqslant u \leqslant 6\varepsilon$, sub-divided into the lethargy intervals $[0, \varepsilon]$, $[\varepsilon, 2\varepsilon], \ldots [5\varepsilon, 6\varepsilon]$, for various moderator masses, ranging from $A = 2$ (deuterium) to $A = 238$ (uranium). The number of iterations required to satisfy the convergence criterion (23) together with the relative accuracy achieved is shown in Table I.

The maximum number of iterations required in any of the lethargy intervals is depicted for both the classical and alternative forms of the integral equation. In addition the relative errors achieved at the last lethargy point $u = 6\varepsilon$ are quoted, comparison having been made with Teichmann's analytical values. The errors achieved throughout the total range $[0, 6\varepsilon]$ are comparable to these and the values at

TABLE I

Maximum Number of Iterations Required and the Relative Accuracy Achieved in the Collision Densities at $u = 6\varepsilon$

	Moderator mass	Classical form (2)–(3)		Alternative form (12)–(13)	
	A	M	E	M	E
$n=4$	2	15	2.9 (−3)	5	3.2 (−7)
	4	10	4.1 (−5)	6	4.3 (−7)
	12	9	1.4 (−5)	8	7.5 (−7)
	16	9	3.5 (−6)	8	9.9 (−7)
	238	8	2.1 (−6)	8	1.8(−6)
$n=8$	2	10	1.1 (−5)	4	1.3(−7)
	4	8	4.7 (−6)	5	2.1(−7)
	12	7	2.9 (−6)	6	7.1 (−7)
	16	7	2.5 (−6)	6	1.1 (−6)
	238	7	2.8 (−6)	7	2.4 (−6)

M = Maximum number of iterations required in any lethargy interval.
E = Relative accuracy in collision density (cf. (23)).
n = Order of quadrature in each lethargy interval.
The lethargy intervals considered are $[0, \varepsilon]$, $[\varepsilon, 2\varepsilon)$,..., $[5\varepsilon, 6\varepsilon]$.

this final point are selected as being representative of the accuracy attained, since they depend on all the previously computed $F(u)$ values for $u < 6\varepsilon$.

It will be seen that in spite of the finite discontinuity at $u = \varepsilon$, rapid convergence is obtained over the complete lethargy range, particularly in the case of the alternative forms (12) and (13) of the integral equation. It is readily verified that this increased rate of convergence is consistent with the usual measure of convergence for the Neumann series provided by the norm defined by [16]

$$\|\mathbf{K}\|_2^2 = \iint |K(u, u')|^2 \, du \, du' \tag{38}$$

Thus, confining attention to the region $R_l = [u_{l-1}, u_l]$ the norm, N_1, corresponding to the kernel $K(u, u') = \gamma \exp(u' - u)$ of Eqs. (2) and (3) is seen to be

$$N_1 = \gamma \sinh \varepsilon \tag{39}$$

independent of l. The corresponding norm, N_2, for the kernel $K(u, u') = \beta$ of equations (12) and (13) is obviously

$$N_2 = \beta\varepsilon. \tag{40}$$

These norms are compared in Table II and it is seen that the variation is mirrored by the behaviour of the number of iterations required, as exhibited in Table 1. This

TABLE II

Comparison of the Norms of the Kernels for Eqs. (2) and (3) and for Eqs. (12) and (13)

A	ε	N_1	N_2
2	2.197	5.000	0.2747
4	1.022	1.889	0.5747
12	0.344	1.198	0.8422
16	0.250	1.142	0.8801
238	0.017	1.008	0.9916

N_1 = Norm for kernel in Eqs. (2) and (3).
N_2 = Norm for kernel in Eqs. (12) and (13).
ε = Lethargy interval width.

confirms that the $\|K\|_2$ norm is a reliable measure of the rate of convergence of the iterative process.

The power of the Clenshaw–Curtis method is demonstrated by the fact that accurate results are obtained according to specification (23) with $E = 10^{-5}$ for orders even as low as $n = 4$ in each lethargy interval, except for the lowest moderator masses. The accuracy attained for the smoother alternative forms (12) and (13) is correspondingly higher and is acceptable over the complete moderator range with this minimal number of quadrature points. For comparison purposes the results of the more accurate quadrature with the order increased to $n = 8$ per interval are shown. There is a corresponding slight decrease in the number of iterations required and acceptable accuracy is now attained from the classical forms (2) and (3) even at low values of A, where the lethargy interval width $\varepsilon = -\ln \alpha$ is largest.

In the present example, since the integral equation is linear, the traditional algebraic approach is available as an alternative method of solution. The integral is replaced by a quadrature at a set of discrete points and the resulting algebraic system of equations is easily solved by forward substitution; see [4].

The presence of the discontinuity in this problem suggests that an equally spaced quadrature rule such as the trapezium rule would be most convenient. Numerical investigation showed that it was necessary to use at least 512 equally spaced grid-points in each of the six lethargy intervals considered to attain an accuracy of $E = 10^{-4}$ only, when the algebraic approach was adopted. This compares most unfavourably with the number of points required by the Clenshaw–Curtis iterative method, as presented in Table I.

More powerful quadrature rules were also employed in an attempt to reduce the number of points used in the algebraic context. Indeed the Clenshaw–Curtis method itself was employed to this end. It was found that even 32 points per interval produced an accuracy of only 10^{-3}. This suggests that the direct iterative solution in conjunction with quadrature procedures produces higher accuracy at less cost than the algebraic utilization of the quadrature process.

(b) *The Equation*

$$\phi(x) = (x+2)^{-2} - \int_0^x G(\phi(t))(x-t+2)^{-2}\,dt. \tag{16}$$

This equation is solved for $0 \leqslant x \leqslant 1$ for the three cases of G mentioned in Section 3, namely: $2\phi(t)$, $2\phi^2(t)$ and $\exp[-\phi(t)]$. The results are presented in Table III.

In each case, to attain a relative accuracy of $E = 10^{-7}$ in the values of $\phi(x)$, it was only necessary to use a Clenshaw–Curtis quadrature rule of order $n = 8$, convergence being achieved after $M = 5$ iterations at most. In fact, for the $G = 2\phi^2$ case, an accuracy of 10^{-8} was attained after only $M = 4$ iterations. The stability and accuracy of the present algorithm were checked by repeating the calculations with $n = 16$ and allowing M to increase to 8.

Comparison was again made with the conventional algebraic approach [4, 7] in which discretization is achieved by replacing the integral in Eq. (16) by a quadrature at the set of points x_i with associated weights w_i. The result is

$$\phi_j = (x_j+2)^{-2} - \sum_{i=0}^{j} w_i (x_j - x_i + 2)^{-2} G_i \tag{41}$$

in which

$$\phi_j = \phi(x_j) \tag{42}$$

and

$$G_j = G(\phi_j), \qquad j = 0, 1, 2, \ldots, n. \tag{43}$$

TABLE III

Solution of Eq. (16) for Various Forms of G

	$G = 2\phi$	$G = 2\phi^2$	$G = \exp(-\phi)$
x	$\phi(x)$	$\phi(x)$	$\phi(x)$
0	0.25	0.25	0.25
0.1	0.215707	0.224092	0.207804
0.2	0.187008	0.202034	0.169654
0.3	0.162869	0.183104	0.134912
0.4	0.142469	0.166741	0.103068
0.5	0.125152	0.152502	0.073711
0.6	0.110389	0.140036	0.046501
0.7	0.097751	0.129060	0.021161
0.8	0.086890	0.119346	−0.002540
0.9	0.077522	0.110708	−0.024798
1.0	0.069411	0.102991	−0.045775

The trapezium rule is once more particularly convenient to apply here with grid points specified according to

$$x_j = jh = j/n \tag{44}$$

and weights given by

$$\begin{aligned} w_j &= h, \qquad j = 1, 2, \ldots, (n-1), \\ w_0 &= w_n = h/2. \end{aligned} \tag{45}$$

For the linear case $G(\phi) = 2\phi$ the set of equations is, of course, linear and the values of the function are readily obtained by forward substitution from the relation

$$\phi_j(1 + h/4) = (1 - h\phi_0)(x_j + 2)^{-2} - 2h \sum_{i=1}^{j-1} \phi_i (x_j - x_i + 2)^{-2} \tag{46}$$

starting with $\phi_0 = 1/4$.

To achieve an accuracy of $E = 10^{-7}$ in the ϕ_j values, which is comparable to the Clenshaw–Curtis precision attained, it was necessary to use a value of $n = 512$. So once again the present algorithm requires considerably fewer points.

Similar remarks apply to the non-linear cases $G = 2\phi^2$ and $G = \exp(-\phi)$ in Eq. (41) where it was again necessary to go to $n = 512$. An additional complication for the algebraic approach here is that it is necessary now to solve a non-linear equation for each ϕ_j to instigate the forward substitution process. Hence, the comparison with Clenshaw–Curtis is even less favourable than in the linear case.

(c) *The Lane–Emden Equation*

As a final illustrative example, calculations were performed on the Lane–Emden differential equation (17) and its integral equation equivalent (18). The cases considered were the linear one with $v = 1$ where the analytic solution $x(t) = \sin t/t$ is available and the non-linear case $v = 5$. In the latter situation the surprisingly simple solution $(1 + t^2/3)^{-1/2}$ is available for checking purposes.

Of particular interest here is the fact that comparison may be made with the standard numerical solutions of the initial value differential equation problem.

The standard method chosen was the powerful Runge–Kutta procedure. For the linear case an accuracy of $E = 1.5 \times 10^{-8}$ was achieved on $0 \leqslant t \leqslant 1$ with a step-size of $h = 0.05$. The calculations were extended to the interval $0 \leqslant t \leqslant 10$ because of the oscillatory nature of the solution and, again, the highly robust Runge–Kutta algorithm achieved an accuracy of $E = 1.1 \times 10^{-8}$ at $t = 10$ with $h = 0.05$. This represents 4×200 function evaluations on the complete interval, since the Runge–Kutta method requires four function evaluations per cycle. Similar results were obtained for the non-linear case $v = 5$. Thus at $t = 1$ with $h = 0.05$ an accuracy of 5.3×10^{-8} was produced, but, when the calculation was extended to $t = 10$, it was necessary to decrease the step-size to $h = 0.025$ and an accuracy of 10^{-9} was then realised.

The present Clenshaw–Curtis iterative procedure was then applied to both cases, utilizing the equivalent integral equation form (18). It was found that an accuracy of 10^{-9} was obtainable over $0 \leqslant t \leqslant 1$ with a quadrature formula whose order was as low as $n = 8$, after only $M = 3$ iterations. Calculations on the extended range $0 \leqslant t \leqslant 10$ produced $E = 10^{-9}$ with $n = 8$ per unit interval (80 points in all) although this was almost certainly an over-estimate of the points required. The number of iterations was again $M = 3$.

So, even here, where a powerful initial value differential equation solver is available, the present method comes into consideration as a possible alternative.

In general, it appears that this Clenshaw–Curtis iterative scheme can provide rapid and accurate solutions of Volterra integral equations of both the linear and the non-linear types.

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